

THE STRUCTURE OF QUASI-MULTIPLIERS OF C^* -ALGEBRAS

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ABSTRACT. Let A be a C^* -algebra and A^{**} its enveloping W^* -algebra. Let $\text{LM}(A)$ be the left multipliers of A , $\text{RM}(A)$ the right multipliers of A and $\text{QM}(A)$ the quasi-multipliers of A . A question was raised by Akemann and Pedersen [1] whether $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$. McKennon [20] gave a nonseparable counterexample. L. Brown [6] shows the answer is negative for stable (separable) C^* -algebras also.

In this paper, we mainly consider σ -unital C^* -algebras. We give a criterion for $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$. In the case that A is stable, we give a necessary and sufficient condition for $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$. We also give answers for other C^* -algebras.

1. INTRODUCTION AND PRELIMINARIES

Definition 1.1. Let A be a C^* -algebra and A^{**} its enveloping von Neumann algebra. An element x in A^{**} is called a multiplier of A if $xa \in A$ and $ax \in A$ for all $a \in A$. Similarly, x is a left multiplier if $xa \in A$, for all $a \in A$, x is a right multiplier if $ax \in A$, for all $a \in A$, and x is a quasi-multiplier if $axb \in A$, for all $a, b \in A$. We denote the sets of multipliers, left multipliers, right multipliers and quasi-multipliers by $\text{M}(A)$, $\text{LM}(A)$, $\text{RM}(A)$ and $\text{QM}(A)$, respectively.

If $\pi: A \rightarrow B(H)$ is a faithful representation, then the extension of π to A^{**} maps $\text{M}(A)$, $\text{LM}(A)$, $\text{RM}(A)$ and $\text{QM}(A)$ isometrically onto the sets of operators in $B(H)$ that satisfy the appropriate multiplication properties relative to $\pi(A)$. Each set $\text{M}(A)$, $\text{LM}(A)$, $\text{RM}(A)$ and $\text{QM}(A)$ is equipped with a natural weak topology.

Definition 1.2. Let A be a C^* -algebra and A^{**} its enveloping von Neumann algebra. The strict topology on A^{**} is generated by the seminorms $x \rightarrow \|xa\|$ and $x \rightarrow \|ax\|$, $a \in A$. Similarly, we have the left strict topology, generated by the seminorms $\|xa\|$, the right strict topology, generated by $\|ax\|$, and the quasi-strict topology, generated by $\|axb\|$, $a, b \in A$.

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$M(A)$ is the strict closure of A , $LM(A)$ is the left strict closure of A , $RM(A)$ is the right closure of A and $QM(A)$ is the quasi-strict closure of A . For detailed expositions of these results the reader is referred to [1, 2, 8 and 21].

$LM(A)$, $RM(A)$ and $QM(A)$ are norm closed subspaces in A^{**} ; $QM(A)$ is $*$ -invariant, whereas $(LM(A))^* = RM(A)$. Moreover, $LM(A)$ and $RM(A)$ are Banach algebras. The best behaved class is $M(A)$ which is a C^* -algebra. It is clear that $M(A) = LM(A) \cap RM(A)$ and that $LM(A) + RM(A) \subset QM(A)$. The question was raised by Akemann and Pedersen [1] in 1973 whether $QM(A) = LM(A) + RM(A)$. McKennon [20] gave a nonseparable counterexample in 1978. Recently, L. Brown showed [6] that even when A is stable and separable, $QM(A)$ may not equal $LM(A) + RM(A)$.

In this paper, we give exact conditions for $QM(A) = LM(A) + RM(A)$ and for $QM(A) \neq LM(A) + RM(A)$.

Definition 1.3. A topological space X is scattered if every closed subset of X has a relatively isolated point.

Definition 1.4. Let X be a scattered topological space. We define $X_{[0]} = X$, $X_{[1]} = X \setminus \{\text{isolated points of } X\}$. If $X_{[\alpha]}$ is defined for some ordinal number α , define $X_{[\alpha+1]} = X_{[\alpha]} \setminus \{\text{isolated points of } X_{[\alpha]}\}$, if β is a limit ordinal, define $X_{[\beta]} = \bigcap_{\alpha < \beta} X_{[\alpha]}$.

Definition 1.5. Let X be a scattered topological space. We define $\lambda(X) = \alpha$, if α is the least ordinal such that $X_{[\alpha]}$ is discrete. Since X is scattered, $\lambda(X)$ is well defined.

Definition 1.6. Let $Y_1 = \{0, 1/n, n = 1, 2, \dots\}$, a subset of $[0, 1]$ with the usual topology, and let Y_2 be the one-point compactification of the disjoint union of countably many copies of Y_1 . If Y_α is defined for some ordinal number α , define $Y_{\alpha+1}$ as the one-point compactification of the disjoint union of countably many copies of Y_α . If β is a limit ordinal, define Y_β as the one-point compactification of the disjoint union of Y_α , $\alpha < \beta$. We also define $Z_\alpha^{(m)}$ to be the union of m disjoint copies of Y_α .

Theorem 1.7 [17] (or see [19, Theorem 1.9]). *Let X be a countable, compact Hausdorff space with $\lambda(X) = \alpha \geq 1$ and assume that $X_{[\alpha]}$ consists of n points. Then X is homeomorphic to $Z_\alpha^{(n)}$.*

Let $\{X, A(t), \mathcal{F}\}$ be a continuous field of C^* -algebras with X a locally compact Hausdorff space. Let $A = C_0(X, A(t), \mathcal{F})$ be the set of all continuous cross sections of $\{X, A(t), \mathcal{F}\}$ vanishing at infinity. Then A is a C^* -algebra.

We say a bounded cross section x in the bundle

$$\{X, LM(A(t))\} \quad (\{X, RM(A(t))\}, \{X, QM(A(t))\})$$

is left-strictly (right-strictly, quasi-strictly) continuous at t_0 , if for every $a \in \mathcal{F}$, xa (ax, axa) is continuous at t_0 . We denote by $C^b(X, LM(A(t))_{L.S.}, \mathcal{F})$

$(C^b(X, \text{RM}(A(t))_{\text{R.S.}}, \mathcal{F}), C^b(X, \text{QM}(A(t))_{\text{Q.S.}}, \mathcal{F}))$ the set of all bounded left-strictly (right-strictly, quasi-strictly) continuous cross sections in

$$\{X, \text{LM}(A(t))\} \quad (\{X, \text{RM}(A(t))\}, \{X, \text{QM}(A(t))\}).$$

Let $A = C_0(X, A(t), \mathcal{F})$. Exactly as in [2, 3.3] we obtain

Theorem 1.8 (see [19, §1.3] also).

$$C^b(X, \text{LM}(A(t))_{\text{L.S.}}, \mathcal{F}) = \text{LM}(A);$$

$$C^b(X, \text{RM}(A(t))_{\text{R.S.}}, \mathcal{F}) = \text{RM}(A);$$

$$C^b(X, \text{QM}(A(t))_{\text{Q.S.}}, \mathcal{F}) = \text{QM}(A).$$

2. A CRITERION FOR $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$

Let A be a σ -unital C^* -algebra, and a a strictly positive element, $0 < a \leq 1$. For each n let f_n be a continuous function such that $f_n(t) = 1$ if $t \geq 1/n$, $f_n(t) = 0$ if $0 \leq t \leq 1/(n+1)$ and f_n is linear in $[1/(n+1), 1/n]$. Define $e_n = f_n(a)$. Then $\{e_n\}$ is an approximate identity for A . Moreover $e_m e_n = e_n e_m = e_n$, if $m > n$.

Lemma 2.1. *Let A be a σ -unital C^* -algebra and $\{e_n\}$ an approximate identity for A satisfying $e_m e_n = e_n e_m = e_n$, if $m > n$. Suppose that $y \in \text{QM}(A)$, then $y \in \text{LM}(A)$ if and only if there exists an increasing sequence $\{n_k\}$ of nonnegative integers such that*

$$\sum_{k=1}^{\infty} (1 - e_{n_{k+1}}) y (e_{n_k} - e_{n_{k-1}})$$

converges in norm to an element of A where $e_{n_0} = 0$.

Proof. Assume that $y \in \text{LM}(A)$. For every m , $y e_m \in A$. Hence there is m' such that $\|(1 - e_{m'}) y e_m\| < 1/2^m$. Therefore we can recursively define $n_1 < n_2 < \dots$ so that

$$\|(1 - e_{n_{k+1}}) y e_{n_k}\| < \frac{1}{2^k}.$$

This implies that $\sum_{k=1}^{\infty} (1 - e_{n_{k+1}}) y (e_{n_k} - e_{n_{k-1}})$ is norm convergent to an element in A .

For the converse, let $z = y - \sum_{k=1}^{\infty} (1 - e_{n_{k+1}}) y (e_{n_k} - e_{n_{k-1}})$. For fixed n , let m be the least integer such that $n_m > n$. Then

$$\begin{aligned} z e_n &= y e_n - \sum_{k=1}^m (1 - e_{n_{k+1}}) y (e_{n_k} - e_{n_{k-1}}) e_n \\ &= \sum_{k=1}^m [y (e_{n_k} - e_{n_{k-1}}) e_n - (1 - e_{n_{k+1}}) y (e_{n_k} - e_{n_{k-1}}) e_n] \\ &= \sum_{k=1}^m e_{n_{k+1}} y (e_{n_k} - e_{n_{k-1}}) e_n. \end{aligned}$$

Since $y \in \text{QM}(A)$, we conclude that $ze_n \in A$, for all n . Hence $z \in \text{LM}(A)$. It follows that

$$y = z + \sum_{k=1}^{\infty} (1 - e_{n_{k+1}}) y (e_{n_k} - e_{n_{k-1}}) \in \text{LM}(A).$$

Lemma 2.2. *Let A be a σ -unital C^* -algebra and $\{e_n\}$ an approximate identity for A satisfying $e_m e_n = e_n e_m = e_n$, if $m > n$. Suppose that $x_n \in \text{QM}(A)$ with $\|x_n\| \leq M$ for some M , j is an integer and $0 < \alpha \leq 1$. Then*

$$\sum_{n=1}^{\infty} (e_{n+j+1} - e_{n+j})^{\alpha} x_n (e_n - e_{n-1})^{\alpha}$$

converges strictly.

Proof. Let P_s be the range projection of $(e_s - e_{s-1})$ and

$$y_s = (e_{s+j+1} - e_{s+j})^{\alpha} x_s (e_s - e_{s-1})^{\alpha}.$$

Clearly $P_s \cdot P_{s+2+i} = 0$ for $i = 0, 1, 2, \dots$. Suppose that $A \subset B(H)$ and $f \in H$. Then

$$\begin{aligned} \left\| \sum_{\substack{s=2k \\ s \leq N}} y_s f \right\|^2 &= \left\| \sum_{\substack{s=2k \\ s \leq N}} P_{s+j+1} y_s P_s f \right\|^2 \\ &= \sum_{\substack{s=2k \\ s \leq N}} \|P_{s+j+1} y_s P_s f\|^2 \leq M^2 \|f\|^2 \end{aligned}$$

for all N . Similarly

$$\left\| \sum_{\substack{s=2k+1 \\ s \leq N}} y_s f \right\|^2 \leq M^2 \|f\|^2 \quad \text{for all } N.$$

So $\{\|\sum_{n=1}^N y_n\|\}$ is bounded. For fixed m , if $N > m + 1$, then

$$e_m \sum_{n=N}^{N+k} y_n = \sum_{n=N}^{N+k} y_n e_m = 0$$

for every k . Hence $\sum_{n=1}^{\infty} y_n$ converges strictly.

Theorem 2.3. *Let A be a σ -unital C^* -algebra and $\{e_n\}$ an approximate identity for A satisfying $e_m e_n = e_n e_m = e_n$, if $m > n$. Then $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$ if and only if for every $x \in \text{QM}(A)_{\text{s.a.}}$, there exists an increasing sequence $\{n_k\}$ of nonnegative integers such that*

$$\sum_{k=1}^{\infty} (1 - e_{n_k}) x (e_{n_k} - e_{n_{k-1}})$$

converges strictly ($e_{n_0} = 0$).

Proof. Let $x \in \text{QM}(A)_{\text{s.a.}}$ and $n_1 < n_2 < \dots$ be chosen such that $\sum_{k=1}^{\infty} (1 - e_{n_k})x(e_{n_k} - e_{n_{k-1}})$ converges strictly. Let $x_k = (1 - e_{n_k})x(e_{n_k} - e_{n_{k-1}})$. Since $\sum_{k=1}^N x_k \in \text{RM}(A)$ for all N , we conclude that $\sum_{k=1}^{\infty} x_k \in \text{RM}(A)$. For a fixed m , suppose that k_0 is the least integer such that $n_{k_0} > m$. Then

$$\begin{aligned} \left(x - \sum_{k=1}^{\infty} x_k\right) e_m &= x e_m - \sum_{k=1}^{k_0} x_k e_m \\ &= \sum_{k=1}^{k_0} e_{n_k} x (e_{n_k} - e_{n_{k-1}}) e_m \in A. \end{aligned}$$

Hence $x - \sum_{k=1}^{\infty} x_k \in \text{LM}(A)$. This implies that $\text{QM}(A)_{\text{s.a.}} \subset \text{LM}(A) + \text{RM}(A)$, and hence $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$.

Next assume that $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$. Equivalently, $\text{QM}(A)_{\text{s.a.}} = \text{ReLM}(A)$. Let $x \in \text{QM}(A)_{\text{s.a.}}$. Thus there is $y \in \text{LM}(A)$ such that $x = y + y^*$. By Lemma 2.1, we can choose $n_1 < n_2 < \dots$ such that the elements $y_k = (1 - e_{n_{k+1}})y(e_{n_k} - e_{n_{k-1}})$ satisfy $\|y_k\| < 2^{-k}$, whence $\sum_{k=1}^{\infty} y_k \in A$. By Lemma 2.2 $\sum_{k=1}^{\infty} (e_{n_{k+1}} - e_{n_k})y(e_{n_k} - e_{n_{k-1}})$ converges strictly. Hence $\sum_{k=1}^{\infty} (1 - e_{n_k})y(e_{n_k} - e_{n_{k-1}})$ converges strictly.

Let

$$\begin{aligned} y_{kj} &= (e_{n_k} - e_{n_{k-1}}) e_{n_{j+1}} y (e_{n_j} - e_{n_{j-1}}), \\ y_k^{(1)} &= (e_{n_k} - e_{n_{k-1}}) e_{n_{k+1}} y (1 - e_{n_k}) (e_{n_k} - e_{n_{k-1}}) \end{aligned}$$

and

$$y_k^{(2)} = (e_{n_k} - e_{n_{k-1}}) e_{n_{k+2}} y (1 - e_{n_k}) (e_{n_{k+1}} - e_{n_k}).$$

Then by Lemma 2.2,

$$\sum_{k=1}^{\infty} y_k^{(1)}, \quad \sum_{k=1}^{\infty} y_k^{(2)} \quad \text{and} \quad \sum_{k=1}^{\infty} \sum_{j=1}^{k+1} y_{kj} = \sum_{k=1}^{\infty} \sum_{j=k+2}^{k+1} y_{kj}$$

converge strictly. Since

$$\sum_{k=1}^{\infty} \sum_{j=k+2}^{\infty} y_{kj} + \sum_{k=1}^{\infty} \sum_{j=1}^{k+1} y_{kj} = \sum_{j=1}^{\infty} e_{n_{j+1}} y (e_{n_j} - e_{n_{j-1}}) = y - \sum_{j=1}^{\infty} y_j.$$

We conclude that $\sum_{k=1}^{\infty} \sum_{j=k+2}^{\infty} y_{kj}$ converges strictly. Thus

$$\begin{aligned} &\sum_{k=1}^{\infty} (e_{n_k} - e_{n_{k-1}}) \left(y - \sum_{j=1}^{\infty} y_j \right) (1 - e_{n_k}) \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (e_{n_k} - e_{n_{k-1}}) e_{n_{j+1}} y (e_{n_j} - e_{n_{j-1}}) (1 - e_{n_k}) \\ &= \sum_{k=1}^{\infty} \sum_{j=k+2}^{\infty} y_{kj} + \sum_{k=1}^{\infty} y_k^{(1)} = \sum_{k=1}^{\infty} y_k^{(2)} \end{aligned}$$

converges strictly. So

$$\sum_{k=1}^{\infty} (1 - e_{n_k}) \left(y^* - \sum_{j=1}^{\infty} y_j^* \right) (e_{n_k} - e_{n_{k-1}})$$

converges strictly. Since $(1 - e_{n_k})(e_{n_j} - e_{n_{j-1}}) = 0$ if $k > j$ and

$$(1 - e_{n_{j+1}})(e_{n_k} - e_{n_{k-1}}) = 0, \quad \text{if } j \geq k,$$

we have

$$\sum_{k=1}^{\infty} (1 - e_{n_k}) \left(\sum_{j=1}^{\infty} y_j \right) * (e_{n_k} - e_{n_{k-1}}) = 0.$$

Finally, since $x = y + y^*$, $\sum_{k=1}^{\infty} x_k$ converges strictly. This completes the proof.

3. LIFTING AND HEREDITARY PROPERTIES

Considering the problem $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$, one may ask the following questions:

(i) If I is an ideal of A such that $\text{QM}(A/I) = \text{LM}(A/I) + \text{RM}(A/I)$ and $\text{QM}(I) = \text{LM}(I) + \text{RM}(I)$, does it follow that $\text{QM}(A) = \text{RM}(A) + \text{LM}(A)$?

(ii) If $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$, does it follow that $\text{QM}(B) = \text{LM}(B) + \text{RM}(B)$ for B in A ?

In this section, we shall show that (i) has a positive answer under a suitable assumption on A , and for some special B 's, (ii) also has a positive answer. However, in general (ii) has a negative answer, as we shall see in Example 8.2.

Theorem 3.1. *Let A be a σ -unital C^* -algebra and B a C^* -subalgebra of A such that the hereditary C^* -subalgebra generated by B is A itself. If $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$, then $\text{QM}(B) = \text{LM}(B) + \text{RM}(B)$.*

Proof. Let $\tilde{A}$ and $\tilde{B}$ be C^* -algebras obtained by adding identities to A and B . Since the hereditary C^* -subalgebra generated by B is A itself, B contains a strictly positive element of A , say a . It follows that A and B share a common approximate identity $\{e_n\}$ satisfying $e_n e_m = e_m e_n = e_n$, if $m > n$. Since e_n converges weakly to the identity of A and the identity of B in A^{**} , $\tilde{A}$ and $\tilde{B}$ have the same identity. Thus

$$\text{QM}(B)_{\text{s.a.}} = [(\tilde{B}_{\text{s.a.}})^m]^- \cap [(\tilde{B}_{\text{s.a.}})_m]^- \subset [(\tilde{A}_{\text{s.a.}})^m]^- \cap [(\tilde{A}_{\text{s.a.}})_m]^- = \text{QM}(A)_{\text{s.a.}}$$

(see [1, Theorem 4.1]). Since A and B have the same approximate identity $\{e_n\}$, we can apply Theorem 2.3 to obtain the desired conclusion.

Let A be a C^* -algebra and I a closed ideal. We shall denote $\text{M}(A) \cap I^{**}$, $\text{LM}(A) \cap I^{**}$, $\text{RM}(A) \cap I^{**}$ and $\text{QM}(A) \cap I^{**}$ by $\text{M}(A, I)$, $\text{LM}(A, I)$, $\text{RM}(A, I)$ and $\text{QM}(A, I)$, respectively. If $x \in \text{M}(A, I)$, and $a \in A$, one can see that ax , $xa \in I$. Moreover, if $x \in \text{LM}(A, I)$, $xa \in I$, etc.

Lemma 3.2. *Let A be a σ -unital C^* -algebra and I an ideal of A . Then $\text{QM}(A, I) = \text{LM}(A, I) + \text{RM}(A, I)$ if one of the following holds:*

- (i) $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$ or
- (ii) $\text{QM}(I) = \text{LM}(I) + \text{RM}(I)$.

Proof. Let $\{e_n\}$ be an approximate identity of A satisfying $e_m e_n = e_n e_m = e_n$, if $m > n$, and $\{u_\lambda\}$ be an approximate identity for I . Let $x \in \text{QM}(A, I)$. If we define $x_{ij} = (e_i - e_{i-1})^{1/2} x (e_j - e_{j-1})^{1/2}$ ($e_0 = 0$), then $x_{ij} \in A \cap I^{**} = I$. There is a subsequence $\{u_n\}$ of $\{u_\lambda\}$ such that

$$\|(1 - u_i)x_{ij}\| < 1/2^{i+j}, \quad j \leq i, \quad i = 1, 2, \dots,$$

and

$$\|x_{ij}(1 - u_i)\| < 1/2^{i+j}, \quad i \leq j, \quad j = 1, 2, \dots.$$

Define $u = \sum_{i=1}^{\infty} (e_i - e_{i-1})^{1/2} u_i (e_i - e_{i-1})^{1/2}$. By Lemma 2.2, $u \in I^{**}$, it is then easy to check that $u \in \text{M}(A, I)$. Define $a_{ji} = (1 - e_{j+1})(e_i - e_{i-1})^{1/2}$, $b_j^{(1)} = (e_j - e_{j-1})^{1+1/2}$, $b_j^{(2)} = (e_{j+1} - e_j)^{1/2}(e_j - e_{j-1})$ and $b_j^{(3)} = (e_{j-1} - e_{j-2})^{1/2}(e_j - e_{j-1})$. Since $a_{ji} = 0$ if $i < j + 1$, we have

$$\begin{aligned} (1 - e_{j+1})[(1 - u)xu](e_j - e_{j-1}) &= \sum_{i=j+1}^{\infty} a_{ji}(1 - u_i)x_{ij}u_j b_j^{(1)} \\ &+ \sum_{i=j+1}^{\infty} a_{ji}(1 - u_i)x_{ij+1}u_{j+1}b_j^{(2)} + \sum_{i=j+1}^{\infty} a_{ji}(1 - u_i)x_{ij-1}u_{j-1}b_j^{(3)}. \end{aligned}$$

Thus

$$\|(1 - e_{j+1})(1 - u)xu(e_j - e_{j-1})\| < 1/2^{j-2}.$$

This implies

$$\sum_{j=1}^{\infty} \|(1 - e_{j+1})(1 - u)xu(e_j - e_{j-1})\| < \infty.$$

By Lemma 2.1, $(1 - u)xu \in \text{LM}(A)$. Similarly, $ux(1 - u) \in \text{RM}(A)$ and $(1 - u)x(1 - u) \in \text{LM}(A) \cap \text{RM}(A)$. For every $a \in A$, $(1 - u)xu \cdot a \in A \cap I^{**} = I$, $a \cdot ux(1 - u) \in I$ and $a(1 - u)x(1 - u)$, $(1 - u)x(1 - u)a \in I$. So $(1 - u)xu \in \text{LM}(A, I)$, $ux(1 - u) \in \text{RM}(A, I)$ and $(1 - u)x(1 - u) \in \text{M}(A, I)$.

Now we need only show that $uxu \in \text{LM}(A, I) + \text{RM}(A, I)$.

(i) If $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$, there are $y_1 \in \text{LM}(A)$ and $z_1 \in \text{RM}(A)$ such that $x = y_1 + z_1$, so $uxu = uy_1u + uz_1u$. Since $u \in \text{M}(A, I)$, $y_1u, ua \in I$ for every $a \in A$. Hence $uy_1u \in \text{LM}(A, I)$. Similarly, $uz_1u \in \text{RM}(A, I)$.

(ii) If $\text{QM}(I) = \text{LM}(I) + \text{RM}(I)$, there are $y_2 \in \text{LM}(I)$ and $z_2 \in \text{RM}(I)$ such that $x = y_2 + z_2$, so $uxu = uy_2u + uz_2u$. One can easily check, as above, that $uy_2u \in \text{LM}(A, I)$, $uz_2u \in \text{RM}(A, I)$. This completes the proof.

At this point, one may ask whether $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$ implies $\text{QM}(I) = \text{LM}(I) + \text{RM}(I)$. This turns out to be false, as we shall see in Example 8.1. However, we have the following “lifting” theorem.

Theorem 3.3. *Let I be an ideal of a σ -unital C^* -algebra A , and suppose that $\text{QM}(A/I) = \text{LM}(A/I) + \text{RM}(A/I)$ and $\text{QM}(I) = \text{LM}(I) + \text{RM}(I)$. Then*

$$\text{QM}(A) = \text{LM}(A) + \text{RM}(A).$$

Proof. Let $\phi: A \rightarrow A/I$ be the canonical homomorphism and take $x \in \text{QM}(A)$. So there is $\bar{y} \in \text{LM}(A/I)$ and $\bar{z} \in \text{RM}(A/I)$ such that $\phi^{**}(x) = \bar{y} + \bar{z}$, where ϕ^{**} is the extension of ϕ to A^{**} . It follows from [6, 4.13] that there are $y_1 \in \text{LM}(A)$, $z_1 \in \text{RM}(A)$ such that $\phi^{**}(y_1) = \bar{y}$ and $\phi^{**}(z_1) = \bar{z}$. Thus $\phi^{**}(x - z_1 - y_1) = 0$. So we may assume that $x \in \ker \phi^{**} \cap \text{QM}(A)$, hence $x \in \text{QM}(A, I)$. By Lemma 3.2 $x \in \text{LM}(A, I) + \text{RM}(A, I) \subset \text{LM}(A) + \text{RM}(A)$.

Let K be the C^* -algebra of all compact operators on l^2 .

Corollary 3.4. *Let A be a C^* -algebra such that $\text{QM}(A \otimes K) = \text{LM}(A \otimes K) + \text{RM}(A \otimes K)$. Then $\text{QM}(\tilde{A} \otimes K) = \text{LM}(\tilde{A} \otimes K) + \text{RM}(\tilde{A} \otimes K)$.*

Theorem 3.5. *Let A be a C^* -algebra such that $\text{QM}(A \otimes K) = \text{LM}(A \otimes K) + \text{RM}(A \otimes K)$ and let B be a σ -unital C^* -subalgebra of A such that $\text{QM}(B) = \text{M}(B)$. Then*

$$\text{QM}(B \otimes K) = \text{LM}(B \otimes K) + \text{RM}(B \otimes K).$$

Proof. By Corollary 3.4 we may assume that A has an identity. Take $x \in \text{QM}(B \otimes K)_{\text{s.a.}}$ and let $\{e_{ij}\}$ be a set of matrix units for K . Then x can be identified with an infinite matrix (a_{ij}) which represents a bounded operator, where a_{ij} is defined by $(1 \otimes e_{ii})x(1 \otimes e_{jj}) = a_{ij} \otimes e_{ij}$. Clearly each $a_{ij} \in \text{QM}(B) = \text{M}(B)$.

Let $\{u_n\}$ be an approximate identity of B which is quasi-central for $\text{M}(B)$, i.e.

$$\lim \|u_n b - b u_n\| = 0 \quad \text{for all } b \in \text{M}(B).$$

For every i , we have an integer n_i such that

$$\max_{k, j \leq i} \left[\left\| (1 - u_{n_i}) u_k \right\| \cdot \|a_{ij}\| \right] < 1/2^{2i+1}$$

and

$$\|u_{n_i} a_{ij} - a_{ij} u_{n_i}\| < 1/2^{i+j}, \quad i \geq j.$$

Let $w = (b_{ij})$, where $b_{ii} = u_{n_i}$, $b_{ij} = 0$, if $i \neq j$. Clearly, w is bounded and so is $wx = (u_{n_i} a_{ij})$. Since $a_{ij} \in \text{M}(B)$, $u_{n_i} a_{ij} \in B \subset A$. We may view wx as an element in $\text{QM}(A \otimes K)$. It follows from [6, 4.20] or Theorem 2.3 that there exist $n_1 < n_2 < \dots$ such that $L(wx)$ is bounded, where

$$L(wx) = \sum_{k=1}^{\infty} (1 - f_{n_k}) wx (f_{n_k} - f_{n_{k-1}})$$

and $f_n = \sum_{i=1}^n 1 \otimes e_{ii}$. Let $\sigma = -L(wx) + L(wx)^*$ and $y = x + \sigma$. Then y is bounded and $\text{Re } y = x$. Let $L(y)' = (c_{ij})$, where $c_{ij} = a_{ij}(1 - u_{n_i})$, if there is $k > l$ such that $n_{k-1} < i \leq n_k$, $n_{l-1} < j \leq n_l$, and $c_{ij} = 0$ otherwise. Then

$L(y) - L(y)' = (d_{ij})$ where $d_{ij} = a_{ij}u_{n_i} - u_{n_i}a_{ij}$ if there is $k > l$ such that $n_{k-1} < i \leq n_k$, $n_{l-1} < j \leq n_l$ and $d_{ij} = 0$ otherwise. Since $\|d_{ij}\| < 1/2^{i+j}$, $d_{ij} \in B$, we see that $L(y) - L(y)' \in B \otimes K$. For every k ,

$$L(y)' \sum_{i=1}^k u_k \otimes e_{ii} \in B \otimes K,$$

because

$$\max_{k, j \leq i} \|a_{ij}\| \cdot \left\| \left(1 - u_{n_i}\right) u_k \right\| < 1/2^{2i+1}.$$

Moreover $[y - L(y)] \cdot \sum_{i=1}^k u_k \otimes e_{ii} \in B \otimes K$. Hence

$$y \cdot \sum_{i=1}^k u_k \otimes e_{ii} \in B \otimes K \quad \text{for all } k.$$

Since $\{\sum_{i=1}^k u_k \otimes e_{ii}\}$ forms an approximate identity for $B \otimes K$, we conclude that $y \in \text{LM}(A)$, so $x \in \text{LM}(A) + \text{RM}(A)$.

4. A CONSTANT ASSOCIATED WITH THE EQUATION

$$\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$$

Definition 4.1. Let A be a C^* -algebra such that $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$. For every $x \in \text{QM}(A)$, let

$$\alpha(x) = \inf\{\|y\| : x = y + z, y \in \text{LM}(A), z \in \text{RM}(A)\}.$$

Clearly $\alpha(x) < \infty$. Let $\alpha(A) = \sup_{\|x\| \leq 1} \alpha(x)$. To see that $\alpha(A) < \infty$, we consider the mapping $\phi : \text{LM}(A) \rightarrow \text{QM}(A)_{\text{s.a.}}$ defined by $\phi(x) = (x + x^*)/2$. Then ϕ is a bounded real linear map from the real Banach space $\text{LM}(A)$ onto the real Banach space $\text{QM}(A)_{\text{s.a.}}$. By the open mapping theorem, ϕ is open. Thus the image of unit ball of $\text{LM}(A)$ under ϕ contains a ball around the origin. It follows that $\alpha(A) < \infty$.

The following is an immediate consequence of Theorem 1.8.

Proposition 4.2. Let A_n be C^* -algebras satisfying $\text{QM}(A_n) = \text{LM}(A_n) + \text{RM}(A_n)$ and $\alpha(A_n) < c$, for some $c > 0$. Then

$$\text{QM}(\Sigma \oplus A_n) = \text{LM}(\Sigma \oplus A_n) + \text{RM}(\Sigma \oplus A_n)$$

and $\sup_n \alpha(A_n) \leq \alpha(\Sigma \oplus A_n) \leq c$.

Lemma 4.3. Suppose that A is a σ -unital C^* -algebra such that $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$. If $\{e_n\}$ is an approximate identity satisfying $e_m e_n = e_n e_m = e_n$, if $m > n$, and $x \in \text{QM}(A)$ with $\|x\| \leq 1$, then for every $\varepsilon > 0$, there is $n_1 < n_2 < \dots$ such that

$$\left\| \sum_{k=1}^{\infty} (1 - e_{n_k}) x (e_{n_k} - e_{n_{k-1}}) \right\| \leq 7\alpha(A) + 5 + \varepsilon.$$

Proof. Let $x = y + z$, where $y \in \text{LM}(A)$, $z \in \text{RM}(A)$ and $\|y\| \leq \alpha(A) + (1/21)\varepsilon$.

As in the proof of Lemma 2.1, there exist $n_1 < n_2 < \dots$ such that

$$\sum_{k=1}^{\infty} \left\| (1 - e_{n_{k+1}}) y (e_{n_k} - e_{n_{k-1}}) \right\| < \frac{\varepsilon}{3}$$

and

$$\sum_{k=1}^{\infty} \left\| (e_{n_k} - e_{n_{k-1}}) z (1 - e_{n_{k+1}}) \right\| < \frac{\varepsilon}{3}.$$

Define $z_{ik} = (e_{n_i} - e_{n_{i-1}}) z (e_{n_k} - e_{n_{k-1}})$. We have

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} (1 - e_{n_k}) z (e_{n_k} - e_{n_{k-1}}) \right\| &= \left\| z - \sum_{k=1}^{\infty} e_{n_k} z (e_{n_k} - e_{n_{k-1}}) \right\| \\ &\leq \|z\| + \left\| \sum_{k=1}^{\infty} \sum_{i=1}^k z_{ik} \right\| \\ &\leq \|z\| + \left\| \sum_{k=1}^{\infty} \sum_{i=1}^{k-2} z_{ik} \right\| + \left\| \sum_{k=1}^{\infty} z_{k-1k} \right\| + \left\| \sum_{k=1}^{\infty} z_{kk} \right\|. \end{aligned}$$

For every large N

$$\begin{aligned} \left\| \sum_{k=1}^N \sum_{i=1}^{k-2} z_{ik} \right\| &= \left\| \sum_{i=1}^{N-2} \sum_{k=i+2}^N z_{ik} \right\| \\ &= \left\| \sum_{i=1}^{N-2} (e_{n_i} - e_{n_{i-1}}) z (e_{n_N} - e_{n_{i+1}}) \right\| < \frac{\varepsilon}{3}. \end{aligned}$$

Thus, by the proof of Lemma 2.2,

$$\left\| \sum_{k=1}^{\infty} (1 - e_{n_k}) z (e_{n_k} - e_{n_{k-1}}) \right\| \leq 5\|z\| + \frac{\varepsilon}{3}.$$

Now we have

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} (1 - e_{n_k}) x (e_{n_k} - e_{n_{k-1}}) \right\| &\leq \left\| \sum_{k=1}^{\infty} (1 - e_{n_{k+1}}) y (e_{n_k} - e_{n_{k-1}}) \right\| \\ &\quad + \left\| \sum_{k=1}^{\infty} (e_{n_{k+1}} - e_{n_k}) y (e_{n_k} - e_{n_{k-1}}) \right\| \\ &\quad + \left\| \sum_{k=1}^{\infty} (1 - e_{n_k}) z (e_{n_k} - e_{n_{k-1}}) \right\| \\ &< \frac{\varepsilon}{3} + 2\|y\| + 5\|z\| + \frac{\varepsilon}{3} \\ &< \frac{\varepsilon}{3} + 2\alpha(A) + \frac{2}{21}\varepsilon + 5\alpha(A) + \frac{5}{21}\varepsilon + 5 + \frac{\varepsilon}{3} \\ &= 7\alpha(A) + 5 + \varepsilon. \end{aligned}$$

Remark 4.4. From the proof of Lemmas 2.2 and 4.3, we find that if the e_n 's are projections, we will have

$$\left\| \sum (1 - e_{n_k}) x (e_{n_k} - e_{n_{k-1}}) \right\| \leq 4\alpha(A) + 3 + \varepsilon.$$

Lemma 4.5. *Let B be a C^* -algebra which has an approximate identity consisting of countably many projections. Suppose that $\text{QM}(B) = \text{M}(B)$ and $A = C(X) \otimes B$, where X is homeomorphic to one of the spaces Y_n described in Definition 1.6. Then*

$$\text{QM}(A) = \text{LM}(A) + \text{RM}(A) \quad \text{and} \quad \alpha(A) \leq \sqrt{n}.$$

Proof. Let $x \in \text{QM}(A) = C(X, \text{QM}(B)_{\text{Q.S.}})$ and $\|x\| \leq 1$. Let $\{e_k\}$ be an approximate identity of B consisting of projections. We use induction.

(1) If $n = 1$, $Y_1 = \{0\} \cup \{1/m\}_{m=1}^\infty$.

For every k , there is an N such that whenever $m \geq N$

$$\|e_k(x(0) - x(1/m))e_k\| < 1/k.$$

There is an integer m_0 such that for every $m \geq m_0$, there is a largest integer k_m such that

$$\|e_{k_m}(x(0) - x(1/l))e_{k_m}\| \leq 1/k_m \quad \text{for all } l \geq m.$$

(In the trivial case $x(1/m) = x(0)$ for all $m \geq m^*$, for some m^* , define $e_{k_m} = e_{m^*}$.) Hence $k_m \rightarrow \infty$, as $m \rightarrow \infty$ and $k_{m+1} \geq k_m$. Define $u(1/m) = e_1$ if $m < m_0$, $u(1/m) = e_{k_m}$ and $y(0) = x(0)$, $y(1/m) = u(1/m)x(1/m)$, $z(0) = 0$ and $z(1/m) = (1 - u(1/m))x(1/m)$. Then $x = y + z$. It is easy to check that for every a and $b \in A$, $y(1/m)a \rightarrow y(0)a$ and $bz(1/m) \rightarrow bz(0)$. So $y \in \text{LM}(A)$, $z \in \text{RM}(A)$ and

$$\|y\| = \|ux\| \leq 1 = \sqrt{1}.$$

(2) Next we assume that Lemma 4.5 is true for all integers less than n . In particular, we can choose $y \in \text{LM}(A)$ such that $\|y\| \leq \sqrt{k}$, where $k < n$.

Notice that Y_n is the one-point compactification of the disjoint union of Z_i , where each Z_i is homeomorphic to Y_{n-1} .

Let $x_i(t) = x(t)|_{Z_i}$. There is an integer i_0 such that for every $i \geq i_0$, there is a largest integer m_i such that $\|e_{m_i}[x(\infty) - x_i(t)]e_{m_i}\| \leq 1/m_i$ for $t \in Z_i$. (In the case that $x_i(t) \equiv x(\infty)$ for all $i \geq i_0$, for some i_0 , we define $e_{m_i} = e_i$.) Hence $m_i \rightarrow \infty$, as $i \rightarrow \infty$ and $m_{i+1} \geq m_i$. By the induction assumption, there are $y_i \in C(Z_i, \text{LM}(B)_{\text{L.S.}})$ and $z_i \in C(Z_i, \text{RM}(B)_{\text{R.S.}})$ such that $x_i = y_i + z_i$ and $\|y_i\| \leq \sqrt{n-1}$.

Define $y(t) = e_{m_i}x_i(t) + (1 - e_{m_i})y_i(t)(1 - e_{m_i})$ if $t \in Z_i$, $y(\infty) = x(\infty)$, $z(t) = (1 - e_{m_i})z_i(t) + (1 - e_{m_i})y_i(t)e_{m_i}$, if $t \in Z_i$ and $z(\infty) = 0$. Clearly, $x = y + z$ and $y(t)|_{Z_i} \in C(Z_i, \text{LM}(B)_{\text{L.S.}})$ and $z(t)|_{Z_i} \in C(Z_i, \text{RM}(B)_{\text{R.S.}})$. Similarly to (1), one can check that $y(t) \in C(Y_n, \text{LM}(B)_{\text{L.S.}}) = \text{LM}(A)$ and

$z(t) \in C(Y_n, \text{RM}(B)_{\text{R.S.}}) = \text{RM}(A)$. Hence $x \in \text{LM}(A) + \text{RM}(A)$. Moreover, let B act on a Hilbert space H and $f \in H$,

$$\begin{aligned} \|y(t)f\|^2 &= \left\| e_{m_i} x_i(t)f \right\|^2 + \left\| (1 - e_{m_i}) y_i(t) (1 - e_{m_i}) f \right\|^2 \\ &\leq \|f\|^2 + (n-1)\|f\|^2 = n\|f\|^2. \end{aligned}$$

This implies that $\|y\| \leq \sqrt{n}$.

Lemma 4.6. Let $A = C_0(X, A(t), A)$ be a separable C^* -algebra, where X is a countable, locally compact Hausdorff space with $\lambda(X) < \infty$ and $A(t)$ are C^* -algebras such that $\text{QM}(A(t)) = \text{M}(A(t))$. Then $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$.

Proof. Let $I_k = \{f \in A, f(t) = 0, \text{ if } t \in X_{[k]}\}$ (cf. Definition 1.4). By Proposition 4.2, and Theorem 3.3, we can easily prove the lemma by induction.

Remark 4.6. Define $f(e^{i\theta}) = i\theta/\pi$ ($-\pi < \theta \leq \pi$) and let a_n ($n \in \mathbb{Z}$) be its Fourier coefficients. Then $a_n = (-1)^{n+1}/n\pi$ ($n \neq 0$), $a_0 = 0$. But $\sum_{n \in \mathbb{Z}} a_{|n|} e^{in\theta}$ is the Fourier series of the L^2 function $2\pi^{-1} \log|1 + e^{i\theta}|$ which is not in $L^\infty(T)$. (T denoting the unit circle.) Thus the matrix (a_{i-j}) represents an operator on l^2 of norm 1. But the lower triangle of the matrix is not bounded. Let $L_n = (b_{ij})$, where $b_{ij} = \sqrt{-1}a_{i-j}$ if $i \geq j$ and $i \leq n$, $b_{ij} = 0$ if $i > n$, or $j > i$.

Let $g \in l^2$, $g = (d_j)$, $d_j = (-1)^j 1/\sqrt{n}^{1/2}$, $j \leq n$, $d_j = 0$, $j > n$. Then $\|g\|_2 = 1$ and

$$\begin{aligned} \|L_n g\|^2 &= \frac{1}{\pi^2 n} \sum_{k=1}^n \left(\sum_{j=1}^{k-1} \frac{1}{j} \right)^2 \geq \frac{1}{\pi^2 n} \sum_{k=1}^n (\log K)^2 \\ &\geq \frac{1}{\pi^2 n} [n \log n (\log n - 2)] \geq \frac{1}{4\pi^2} (\log n)^2 \end{aligned}$$

if n is large enough ($n > 15$). We conclude that $\|L_n\| \geq (1/2\pi) \log n$ when n is large.

Lemma 4.7. Let $A_n = C(Y_n) \otimes K$. Then

$$\frac{1}{9\pi} \log n \leq \alpha(A_n) \leq \sqrt{n}$$

when n is large enough ($n > 20$).

Proof. It follows from Lemma 4.5 that $\text{QM}(A_n) = \text{LM}(A_n) + \text{RM}(A_n)$ and $\alpha(A_n) \leq \sqrt{n}$.

For every sequence $\{n_k\}$, $n_1 < n_2 < \dots$, define the operator $\alpha(\{n_k\}) = (t_{ij})$ where

$$t_{ij} = \begin{cases} \sqrt{-1}(a_{k-l}), & \text{if } i = n_k, j = n_l, \\ 0 & \text{otherwise} \end{cases}$$

and $a_n = (-1)^{n+1}/n\pi$, $n \neq 0$, $n \in \mathbb{Z}$, $a_0 = 0$.

Then $\alpha(\{n_k\})$ is selfadjoint and $\|\alpha(\{n_k\})\| \leq 1$. For every n , define $\alpha_n(\{n_k\}) = (t'_{ij})$, where

$$t'_{ij} = \begin{cases} \sqrt{-1}(a_{k-l}), & \text{if } i = n_k, j = n_l \text{ and } k, l \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Let $S_n = \{a_n(\{n_k\}), \{n_k\} \subset \mathbb{N}\}$, so that each S_n is countable ($n \in \mathbb{N}$). We claim that the cluster points of S_n are

$$\{\alpha_j(\{n_k\}), j \leq n-1, \{n_k\} \subset N\} = \bigcup_{j=0}^{n-1} S_j$$

(in the weak operator topology).

Let $p_m = (\varepsilon_{ij})$ where $\varepsilon_{ii} = 1$, if $i \leq m$, $\varepsilon_{ij} = 0$, if $i \neq j$ or $i > m$.

Let $\beta \in \bigcup_{j=0}^{n-1} S_j$, say $\beta = \alpha_j(\{n_k\})$ for some $j \leq n-1$ and $\{n_k\} \subset \mathbb{N}$. Define $n_k^{(s)} = n_k$ if $k \leq j$, $n_k^{(s)} = n_k + k + s$, if $k > j$. For every m , if $s > m$, $p_m[\beta - \alpha_n(\{n_k^{(s)}\})]p_m = 0$. This implies that $\alpha_n(\{n_k^{(s)}\}) \rightarrow \beta$ weakly as $s \rightarrow \infty$.

Next let $\alpha_n(\{m_k^{(s)}\}) \rightarrow \beta$ weakly as $s \rightarrow \infty$. Since for every i there are only finitely many different elements $p_i \alpha_n(\{m_k^{(s)}\}) p_i$, we see that

$$p_i[\alpha_n(m_k^{(s)}) - \beta]p_i = 0,$$

when s is large. Thus we conclude that

$$\beta \in \{\alpha_j(\{n_k\}), j \leq n-1, \{n_k\} \subset N\} = \bigcup_{j=0}^{n-1} S_j$$

unless $\alpha_n(\{n_k^{(s)}\}) = \beta$ for $s \geq s_0$ for some s_0 . This establishes the claim.

By induction, we have $(S_n)^w = \bigcup_{j=1}^n S_j$ (where “ w ” means the weak closure) and $(S_n)_{[i]}^w = \bigcup_{j=0}^{n-i} S_j$. Hence $\lambda(S_n^w) = n$. It is also clear, by a similar argument as the above, that every sequence of S_n has a convergent subsequence. So S_n^w is compact.

The weak operator topology on bounded subsets of $B(l^2)$ coincides with the quasi-strict topology $B(l^2) = \text{QM}(K)$, so we define a continuous mapping F_n from Y_n onto S_n^w , then $F_n \in C(Y_n, \text{QM}(K)_{\text{Q.S.}}) = \text{QM}(A_n)$. The existence of F_n comes from Theorem 1.7. Now let $\{e_{ij}\}$ be a set of matrix units for K and $f_m = \sum_{i=1}^m 1 \otimes e_{ii}$ (f_m can be identified with a constant function: $Y_n \rightarrow P_m \in K$). Then $\{f_m\}$ forms an approximate identity for A . For every $n_1 < n_2 < \dots$, we see clearly, by the construction of S_n^w , that

$$\left\| \sum_{k=1}^{\infty} (1 - f_{n_k}) F_n (f_{n_k} - f_{n_{k-1}}) \right\| \geq \|L_n(\alpha)\| \geq \frac{1}{2\pi} \log n,$$

if n is large enough ($n > 15$).

Combining this and Remark 4.4, we have that $\alpha(A_N) \geq \frac{1}{9\pi} \log n$, if n is large enough ($n > 20$).

We shall see in Example 8.1 that $M(A/I) = QM(A/I)$ and $M(I) = QM(I)$ does not imply $QM(A) = M(A)$ even for very simple algebras. Theorem 3.3 shows that if moreover A is σ -unital, we do have $QM(A) = LM(A) + RM(A)$. Hence the only significance of the following proposition is the estimate of $\alpha(A)$.

Proposition 4.8. *Let A be a σ -unital C^* -algebra and I an ideal of A . Suppose $M(A/I) = QM(A/I)$ and $M(I) = QM(I)$. Then for every $x \in QM(A)$, there are $y \in LM(A)$ and $z \in RM(A)$ such that $x = y + z$, $\|y\| \leq \|x\|$ and $\|z\| \leq \|x\|$, hence $\alpha(A) \leq 1$.*

Proof. Let $\phi : A \rightarrow A/I$ be the canonical homomorphism, ϕ^{**} be its extension to A^{**} . So $\phi(x) \in QM(A/I) = M(A/I)$. By [26], there is $x' \in M(A)$ such that $\phi(x') = \phi(x)$ and $x_1 = x - x' \in \ker \phi^{**} \cap QM(A)$. Thus $x_1 \in QM(A, I)$. Let $\{e_i\}$ be an approximate identity for A satisfying $e_i e_j = e_i$, if $i < j$, and put $x_{ij} = (e_i - e_{i-1})^{1/2} x (e_j - e_{j-1})^{1/2}$. Then $x_{ij} \in I$. Suppose that $\{u_\lambda\}$ is an approximate identity for I . There is a subsequence $\{u_{\lambda_n}\}$ of $\{u_\lambda\}$ such that

$$\left\| (1 - u_{\lambda_i}) x_{ij} \right\| < \frac{1}{2^{i+j}}, \quad i \geq j, \quad i = 1, 2, \dots$$

If $u = \sum_{i=1}^{\infty} (e_i - e_{i-1})^{1/2} u_{\lambda_i} (e_i - e_{i-1})^{1/2}$, then $u \in M(A, I)$. As in the proof of Lemma 3.2 we have

$$\sum_i \left\| (1 - e_{i+1})(1 - u)x_1(e_i - e_{i-1}) \right\| < \infty.$$

Hence $(1 - u)x_1 \in LM(A)$ by Lemma 2.1.

Let $y = (1 - u)x = (1 - u)x' + (1 - u)x_1$ and put $z = ux = ux_1 + ux'$. Since $u \in M(A, I)$, we see that $u \in LM(A)$ and $ux' \in M(A)$. Since $M(I) = QM(I)$, $x_1 \in M(I)$. For every $a \in A$, $au \in I$, so $aux_1 \in I \subset A$. This implies that $ux_1 \in RM(A)$. Hence $z \in RM(A)$ since $0 \leq u \leq 1$. We have $\|y\| = \|(1 - u)x\| \leq \|x\|$ and $\|z\| = \|ux\| \leq \|x\|$. Thus $\alpha(A) \leq 1$.

5. THE SPECTRUM OF AN ELEMENT IN A SCATTERED C^* -ALGEBRA

In this section, we shall discuss the relationship between the spectrum of a single element in a scattered C^* -algebra A and the spectrum of the algebra A .

Jensen [13] defined a C^* -algebra to be scattered if every state on the algebra is atomic. He showed [14] that a C^* -algebra is scattered if and only if it is type I and has scattered spectrum $\widehat{A}$. He also showed [14] that a C^* -algebra is scattered if and only if it has a composition series with elementary quotients.

We recall that a C^* -algebra A is AF (approximately finite-dimensional) if for each $\varepsilon > 0$ and $a_1, a_2, \dots, a_n \in A$ there is a C^* -subalgebra B of A and $b_1, b_2, \dots, b_n \in B$ such that B is of finite dimension and $\|a_i - b_i\| < \varepsilon$, for all $i = 1, 2, \dots, n$.

Lemma 5.1. *Every scattered C^* -algebra A is AF.*

Proof. Suppose that A has a series of ideals $0 = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_\alpha \subset \cdots \subset I_\lambda = A$, where each $I_{\beta+1}/I_\beta$ is an elementary C^* -algebra and $I_\alpha = (\bigcup_{\beta < \alpha} I_\beta)^-$ for each limit ordinal α . We prove the lemma by induction on λ . Assume Lemma 5.1 is true for all $\lambda < \lambda_0$.

If λ_0 is not a limit ordinal, $I_{\lambda_0}/I_{\lambda_0-1}$ is an elementary C^* -algebra, hence $I_{\lambda_0}/I_{\lambda_0-1}$ is an AF-algebra. By the induction hypothesis I_{λ_0-1} is also an AF-algebra. It follows from [11] that A is an AF-algebra.

If λ_0 is a limit ordinal, A is the norm closure of $\bigcup_{\lambda < \lambda_0} I_\lambda$. For each $\varepsilon > 0$ and $a_1, a_2, \dots, a_n \in A$, there is $\lambda < \lambda_0$ and $b'_1, b'_2, \dots, b'_n \in I_\lambda$ such that

$$\|a_i - b_i\| < \varepsilon/2, \quad i = 1, 2, \dots, n.$$

Since, by hypothesis, I_λ is an AF-algebra, there is a C^* -subalgebra B of I_λ and $b_1, b_2, \dots, b_n \in B$ such that B is of finite dimension and

$$\|b'_i - b_i\| < \varepsilon/2, \quad i = 1, 2, \dots, n.$$

Hence $\|b_i - a_i\| < \varepsilon$, $i = 1, 2, \dots, n$. So A is an AF-algebra.

Lemma 5.2. *Let A be a scattered C^* -algebra. If $\lambda(\hat{A}) = \alpha$, then for every $a \in A_{\text{s.a.}}$, we have $\lambda[\sigma(a)] \leq \alpha + 1$. If α is a limit ordinal and $\hat{A}_{[\alpha]} = \emptyset$, then $\lambda[\sigma(a)] \leq \alpha$.*

Proof. Let $I_i = \{x \in A; \pi(x) = 0, \forall \pi \in \hat{A}_{[i]}\}_{i \leq \alpha}$. Suppose that $\alpha \in A_{\text{s.a.}}$. Let B be the C^* -algebra generated by a . Define $J_i = B \cap I_i$. Clearly, since I_{i+1}/I_i and A/I_α are dual C^* -algebras [12, 4.7.20], so are J_{i+1}/J_i and B/J_α . Thus $\hat{B}$ is the union of closed subsets X_i satisfying $X_i \supset X_{i+1}$, $X_{i+1} \subset (X_i)_{[1]}$ and $X_{i+1} \setminus X_i$ is discrete, $i < \alpha$. If $\hat{A}_{[\alpha]} = \emptyset$, $A = I_\alpha$. Hence $X_\alpha = \emptyset$. Since $B = C_0(\hat{B})$, it is clear that $\lambda(\hat{B}) \leq \alpha$ and if $\hat{A}_{[\alpha]} = \emptyset$, $\hat{B}_{[\alpha]} = \emptyset$. Thus $\lambda(\sigma(a)) \leq \alpha + 1$ and if $\hat{A}_{[\alpha]} = \emptyset$, $\lambda(\sigma(a)) \leq \alpha$.

Lemma 5.3. *Let A be a scattered C^* -algebra. Suppose that $\lambda(\hat{A}) = \alpha$, $I_\beta = \{x \in A; \pi(x) = 0, \pi \in \hat{A}_{[\beta]}\}$. Then $I_{\beta+1}/I_\beta$ is of infinite dimension, if $\beta < \alpha$.*

Proof. We shall use the facts that A is of type I and $\beta + 1 \leq \alpha$.

Let $J_{\beta+1} = I_{\beta+1}/I_\beta$. If $\hat{J}_{\beta+1}$ is an infinite set, the result is clear. We may assume therefore that $\hat{J}_{\beta+1} = \{\tilde{\pi}_1, \tilde{\pi}_2, \dots, \tilde{\pi}_m\}$. Let π_i be an irreducible representation of A corresponding to $\tilde{\pi}_i$. We have $\hat{A}_{[\beta]} = \bigcup_{i=1}^m \{\pi_i\}^-$. Since $\hat{A}_{[\beta+1]} \neq \emptyset$, there is $\pi \in \hat{A}_{[\beta+1]}$ and hence there is $i \leq m$ such that $\ker \pi_i \subset \ker \pi$. This implies that $\pi_i(A)$ must be infinite dimensional. Hence $\pi_i(I_{\beta+1}) \supset K(H_{\pi_i})$ (the compact operators on H_{π_i}), where $\dim H_{\pi_i} = \infty$. Since $\pi_i(I_\beta) = 0$, we conclude that $J_{\beta+1}$ is of infinite dimension.

Theorem 5.4. *Let A be a scattered C^* -algebra with $\lambda(\hat{A}) = \alpha$. Then*

- (i) *For every $a \in A_{\text{s.a.}}$ $\lambda(\sigma(a)) \leq \alpha + 1$.*

- (ii) If α is not a limit ordinal, there is $a \in A_{\text{s.a.}}$ such that $\lambda(\sigma(a)) \geq \alpha$.
- (iii) If α is not a limit ordinal, then there is $a \in A_{\text{s.a.}}$ such that $\lambda(\sigma(a)) = \alpha + 1$ if and only if A/I_α is of infinite dimension, where $I_\alpha = \{x \in A; \pi(x) = 0, \pi \in \widehat{A}_{[\alpha]}\}$.
- (iv) If α is a limit ordinal and A/I_α is of finite dimension (or zero), then for every $a \in A_{\text{s.a.}}$, $\lambda(\sigma(a)) \leq \alpha$. Moreover, for every $\beta < \alpha$, there is $a \in A_{\text{s.a.}}$ such that $\lambda(\sigma(a)) > \beta$.
- (v) If α is a limit ordinal such that $\alpha = \lim \beta_n$ ($\beta_n < \alpha$) and A/I_α is of infinite dimension, then there is $a \in A_{\text{s.a.}}$ such that $\lambda(\sigma(a)) \geq \alpha$.

Proof. We shall use induction.

Assume the theorem is true for all $\beta < \alpha$.

(i) is the same as Lemma 5.4.

(ii) If α is not a limit ordinal, by Lemma 5.5, $I_\alpha/I_{\alpha-1}$ is of infinite dimension. By the induction hypothesis for (iii), there is $a \in I_\alpha$ such that a is selfadjoint and $\lambda(\sigma(a)) \geq (\alpha - 1) + 1 = \alpha$.

(iii) If A/I_α is of finite dimension, $a \in A_{\text{s.a.}}$, then there is a polynomial $p(t)$ ($p \neq 0$) such that $p(a) \in I_\alpha$. By the induction hypothesis $\lambda(\sigma(p(a))) \leq \alpha$, since $\lambda(\widehat{I}_\alpha) = \alpha - 1$. By the spectral mapping theorem, one sees easily that $\lambda(\sigma(a)) \leq \alpha$.

If A/I_α is of infinite dimension, there is a sequence of mutually orthogonal projections $\bar{p}_n \in A/I_\alpha$, $\bar{p}_n \neq 0$. Let $\phi : A \rightarrow A/I_\alpha$ be the canonical homomorphism. Since I_α is an AF-algebra, by the projection lifting theorem [4], there is $p_1 \in A$ such that $\phi(p_1) = \bar{p}_1$. Using the projection lifting theorem on $(1 - p_1)A(1 - p_1)/I_\alpha \cap (1 - p_1)A(1 - p_1) \cong (1 - \bar{p}_1)(A/I_\alpha)(1 - \bar{p}_1)$, and continuing, we construct a sequence of mutually orthogonal projections $\{p_n\} \subset A$ such that $\pi(p_n) = \bar{p}_n$. Since there is $\pi \in \widehat{A}_{[\alpha]}$ such that $\pi(p_n) \neq 0$, we have $\pi \in \widehat{A} \setminus \text{hull}(p_n A p_n)$. It follows from the fact that $p_n A p_n$ is a hereditary C^* -subalgebra of A that $(p_n A p_n)^\wedge$ is homeomorphic to $\widehat{A} \setminus \text{hull}(p_n A p_n)$. Since $(\widehat{A} \setminus \text{hull}(p_n A p_n))$ is open and $\widehat{A}_{[\alpha]} \cap (\widehat{A} \setminus \text{hull}(p_n A p_n)) \neq \emptyset$, $\lambda((p_n A p_n)^\wedge) = \alpha$. By (ii), there are $a_n \in p_n A p_n$, $a_n = a_n^*$, $\|a_n\| \leq 1$ and $\lambda(\sigma(a_n)) \geq \alpha$. Taking a_n^2 , if necessary, we may assume that $0 \leq a_n \leq 1$. Define

$$a = \sum_{n=1}^{\infty} \frac{1}{2^n} (p_n + a_n);$$

then a is selfadjoint and $\lambda(\sigma(a)) = \alpha + 1$.

(iv) Assume that α is a limit ordinal and A/I_α is of finite dimension. If $a \in I_\alpha$, $a = a^*$, then by Lemma 5.2 $\lambda(\sigma(a)) \leq \alpha$. For every $a \in A_{\text{s.a.}}$, there is a polynomial $p(t)$ ($p(t) \neq 0$) such that $p(a) \in I_\alpha$. Hence $\lambda(\sigma(p(a))) \leq \alpha$. By the spectral mapping theorem, one can see easily that $\lambda(\sigma(a)) \leq \alpha$. For each $\beta < \alpha$, consider $I_{\beta+1} \subset A$. By the induction hypothesis, there is $a \in A_{\text{s.a.}}$ such that $\lambda(\sigma(a)) \geq \beta$.

(v) If α is a limit ordinal such that $\alpha = \lim \beta_n$, $\beta_n < \alpha$, and A/I_α is of infinite dimension, then, as in the proof of (iii), A contains a sequence of mutually orthogonal projections $\{q_n\}$ such that $\lambda[(q_n A q_n)^\wedge] = \alpha$. By (iv), there are $a_n \in q_n A q_n$, $0 \leq a_n \leq 1$ such that, $\lambda(\sigma(a_n)) \geq \beta_n$. Define

$$a = \sum_{n=1}^{\infty} \frac{1}{2^n} (q_n + a_n).$$

Clearly $a \in A_{\text{s.a.}}$ and $\lambda(\sigma(a)) \geq \alpha$.

The proof is complete.

6. QUASI-MULTIPLIERS OF STABLE C^* -ALGEBRAS

Lemma 6.1. *Let A be a separable scattered C^* -algebra with $\lambda(\hat{A}) < \infty$. Then $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$.*

Proof. Let $I_i = \{a \in A, \pi(a) = 0, \forall \pi \in \hat{A}_{[i]}\}$. Then $\{0\} = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n \subset A$, $n = \lambda(\hat{A})$, and I_i/I_{i-1} and A/I_n are separable dual C^* -algebras. Since A and I_i are σ -unital and $\text{M}(I_i/I_{i-1}) = \text{QM}(I_i/I_{i-1})$, $\text{M}(A/I_n) = \text{QM}(A/I_n)$, by Theorem 3.3 and induction; $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$.

Corollary 6.2. *Let A be a separable scattered C^* -algebra with $\lambda(\hat{A}) < \infty$. Then $\text{QM}(A \otimes K) = \text{LM}(A \otimes K) + \text{RM}(A \otimes K)$.*

Theorem 6.3. *Let A be a separable C^* -algebra. Then $\text{QM}(A \otimes K) = \text{LM}(A \otimes K) + \text{RM}(A \otimes K)$ if and only if A is scattered and $\lambda(\hat{A}) < \infty$.*

Proof. By Corollary 6.2, we need only show the “only if” part. So we assume that $\text{QM}(A \otimes K) = \text{LM}(A \otimes K) + \text{RM}(A \otimes K)$. It follows from Corollary 3.4 that we may assume that A has an identity. It follows from [6, 4.23] that A is scattered. If $\lambda(\hat{A})$ is not finite, by Theorem 5.4, for every integer $m > 0$, there is $a \in A_{\text{s.a.}}$ such that $\lambda(\sigma(a)) = m$. Let B be the C^* -algebra generated by a and 1. It follows from the proof of Proposition 2.4 that $\text{QM}(B \otimes K) \subset \text{QM}(A \otimes K)$ and $A \otimes K$ and $B \otimes K$ share a common approximate identity $f_n = \sum_{i=1}^n 1 \otimes e_{ij}$, where $\{e_{ij}\}$ is a set matrix units for K . By Lemma 2.16, there is $F \in \text{QM}(B \otimes K) \subset \text{QM}(A \otimes K)$ such that for every $\{n_k\}$, $n_1 < n_2 < \cdots$,

$$\left\| \sum_{k=1}^{\infty} (1 - f_{n_k}) F (f_{n_k} - f_{n_{k-1}}) \right\| \geq \frac{1}{2\pi} \log m,$$

if m is large enough. It follows from Lemma 4.3 and Remark 4.4 that

$$\alpha(A \otimes K) \geq \frac{1}{9\pi} \log m$$

for m large enough. Hence $\alpha(A \otimes K) = \infty$, a contradiction.

Corollary 6.4. *Let A be a separable C^* -algebra. Then $\text{QM}(A \otimes K) = \text{LM}(A \otimes K) + \text{RM}(A \otimes K)$ if and only if there is an integer $m > 0$ such that for every $a \in A_{\text{s.a.}}$, $\sigma(a)$ is countable and $\lambda(\sigma(a)) \leq m$.*

Proof. It is an immediate consequence of Theorem 5.1, [14, Theorem 2.2] and Theorem 6.2.

Corollary 6.5. *Let A be a separable stable C^* -algebra. Then $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$ if and only if A is scattered and $\lambda(\hat{A}) < \infty$.*

7. C^* -ALGEBRAS WITH FINITE DIMENSIONAL IRREDUCIBLE REPRESENTATIONS

In this section we shall consider C^* -algebras whose irreducible representations are finite dimensional. Let M_n denote the C^* -algebra of all complex $n \times n$ matrices. If A is a C^* -algebra whose irreducible representations are finite dimensional and $\hat{A}$ is Hausdorff, then by [9, Theorem 10.54], $A = C_0(\hat{A}, M_{n(t)}, A)$. If $A = C_0(\hat{A}, M_{n(t)}, A)$ is locally trivial, one can easily show by Theorem 1.3 that $\text{QM}(A) = \text{M}(A)$. However, even if $\hat{A}$ is countable and Hausdorff, $\text{QM}(A) \neq \text{LM}(A) + \text{RM}(A)$, in general.

Proposition 7.1. *There is a C^* -algebra A such that all of its irreducible representations are finite dimensional, $\hat{A}$ is a countable locally compact Hausdorff space, and $\text{QM}(A) \neq \text{LM}(A) + \text{RM}(A)$.*

Proof. Keep the notations in the proof of Lemma 4.5. Let $P^{(n)}(t)$ be the range projection of $F_n(t)$. By the proof of Lemma 4.5, it is clear that $P^{(n)}(t)$ is a weakly continuous mapping from Y_n to K . Since $P^{(n)}(t)$ is bounded, we conclude that $P^{(n)}(t) \in \text{QM}(C(Y_n, K))$.

Let X be the disjoint union of Y_n , $n = 1, 2, \dots$. Define

$$B_0 = \{x \in C_0(X, K) : x(t) = P^{(n)}(t)x(t)P^{(n)}(t); \forall t \in Y_n\}.$$

Clearly, B_0 is a $*$ -algebra. Let $M_n(t) = P^{(n)}(t)KP^{(n)}(t)$. Then each $M_n(t)$ is isomorphic to some M_k . We define $A = C_0(X, M_n(t), B_0)$. A is a C^* -algebra all of whose irreducible representations are of finite dimension and $\hat{A} = X$, a countable, locally compact Hausdorff space. Define

$$q_k(t) = P^{(n)}(t) \sum_{i=1}^k 1 \otimes e_{ii} P^{(n)}(t), \quad \text{if } k \geq m(n) \text{ and } t \in Y_n,$$

and $q_k(t) = 0$, if $k < m(n)$ and $t \in Y_n$, where $m(n)$ is the largest integer such that

$$\|L_{m(n)}(\alpha)\| \leq [\log(n+1)]^{1/4}.$$

Since $m(n) \rightarrow \infty$, $q_k(t) \in A$. Moreover, $\{q_k(t)\}$ forms an approximate identity for A .

Define $F(t) = F_n(t)$, if $t \in Y_n$, $n = 1, 2, \dots$, so that $F(t) \in \text{QM}(A)$. By the proof of Lemma 4.5 we have for every $\{n_k\} \subset \mathbb{N}$, if n is large, that

$$\begin{aligned} & \left\| \sum \left(1 - q_{n_k}\right) F \left(q_{n_k} - q_{n_{k-1}}\right) \right\| \\ & \geq \frac{1}{2\pi} \log n - [\log(n+1)]^{1/4} \quad (\rightarrow \infty, \text{ as } n \rightarrow \infty). \end{aligned}$$

Thus $F \notin \text{LM}(A) + \text{RM}(A)$.

Theorem 7.2. *Let A be a σ -unital C^* -algebra whose dimensions of irreducible representations are bounded by an integer n . Then*

$$\text{QM}(A) = \text{LM}(A) + \text{RM}(A).$$

Proof. We shall use induction on n .

Assume that Theorem 7.2 is true for all $n \leq k$. Let $n = k + 1$ and $I = \{x \in A : \pi(x) = 0, \text{ if } \dim \pi \leq k\}$. By [21, 4.4.10], I is an ideal. Moreover, I is a homogeneous C^* -algebra of order $n = k + 1$. So I arises from a locally trivial M_{k+1} -bundle [12]. Hence $\text{QM}(I) = \text{M}(I)$. Now A/I is a σ -unital C^* -algebra whose irreducible representations have dimensions bounded by k . By the induction hypothesis, $\text{QM}(A/I) = \text{LM}(A/I) + \text{RM}(A/I)$. It follows from Theorem 3.3 that $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$.

Akemann and Shultz showed in [3] that a type I C^* -algebra is perfect if and only if every convergent sequence in $\hat{A}$ converges to at most a countable number of points. So the algebras in Proposition 7.1 and Theorem 7.2 are perfect. We shall produce an imperfect C^* -algebra A , such that all of its irreducible representations are finite dimensional and $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$.

Example 7.3. Let H be a separable infinite dimensional Hilbert space and $\{H_n\}$ a sequence of mutually orthogonal, infinite dimensional subspaces. Let e_n be the projection corresponding to H_n . There are sequences of finite rank projections $\{p_i^n\}$ together with a collection $\{q_\sigma^n\}$ of infinite rank projections indexed by binary strings σ of 0's and 1's such that

- (i) $\sum p_i^n = e_n$ for each n ,
- (ii) $p_i^n q_\sigma^n = q_\sigma^n p_i^n$ for all i, σ and n ,
- (iii) $q_0^n + q_1^n = e_n$ for each n ,
- (iv) $q_{\sigma_0}^n + q_{\sigma_1}^n = q_\sigma^n (e_n - p_m^n)$ for all σ , where $m = |\sigma|$ (see [3, Proposition 3.14]).

Let I be the C^* -algebra of all compact operators on H which commute with $\{p_i^{(n)}\}$. Let A be the C^* -algebra generated by I and by the set of projections $\{q_\sigma^n\}$.

We claim that A is an imperfect, separable C^* -algebra all of whose irreducible representations are finite dimensional (and without identity). Clearly I is an ideal of A . Moreover, I is the restricted directed sum of finite dimensional ideals of A . Since the q_σ^n 's commute with each other, A/I is abelian. It follows that every irreducible representation of A is finite dimensional. By [3, Proposition 3.14], A is not perfect. By Theorem 3.3, $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$.

8. EXAMPLES

Example 8.1. $\text{QM}(A/I) = \text{M}(A/I)$ and $\text{QM}(I) = \text{M}(I)$, but $\text{QM}(A) \neq \text{M}(A)$.

Let A be the C^* -algebra of convergent sequences in M_2 with limits of the form $\begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}$. Then it is easy to see that $\text{QM}(A)$ consists of those bounded

sequences $\{x_n\}_{n=1}^\infty$ in M_2 such that $(x_n)_{11} \rightarrow (x_\infty)_{11}$, whereas $M(A)$ consists of those bounded sequences $\{x_n\}_{n=1}^\infty$ in M_2 such that $(x_n)_{11} \rightarrow (x_\infty)_{11}$, $(x_n)_{21} \rightarrow 0$ and $(x_n)_{12} \rightarrow 0$. Thus $QM(A) \neq M(A)$.

Let I be the ideal of A consisting of sequences $\{x_n\}_{n=1}^\infty$ in M_2 such that $x_n \rightarrow 0$. Then $QM(I) = M(I)$. Since A/I is one dimensional, $QM(A/I) = M(A/I)$.

Example 8.2. $QM(A) \neq M(A)$, $QM(A) = LM(A) + RM(A)$ but $QM(I) \neq LM(I) + RM(I)$.

Let x be a countable compact Hausdorff space with $\lambda(x) = \omega$, where ω is the first limit ordinal. Let $\{e_{ij}\}$ be a set of matrix units for K .

Suppose $B_0 = C(x) \otimes K$, $B = \tilde{B}_0$. Let A be the C^* -algebra of convergent sequences in B with limits in $C(x) \otimes e_{11}$. We identify $x \in B_0$ with an infinite matrix (a_{ij}) , where $a_{ij} \in C(x)$ is defined by $(1 \otimes e_{ii})x(1 \otimes e_{jj}) = a_{ij} \otimes e_{ij}$. Identifying the identity of $\tilde{B}_0$ with the identity matrix, we can identify elements of $\tilde{B}_0$ with some infinite matrices. It is easy to check (by Theorem 1.4, for example) that $QM(A)$ consists of these bounded sequences $\{(a_{ij}^{(n)})\}_{n=1}^\infty$ in B such that $a_{11}^{(n)} \rightarrow a_{11}^\infty$ and $M(A)$ consists of those bounded sequences $\{(a_{ij}^{(n)})\}_{n=1}^\infty$ in B such that $a_{11}^{(n)} \rightarrow a_{11}^\infty$ and $a_{ij}^{(n)} \rightarrow 0$, if $i \cdot j \neq 1$, clearly $QM(A) \neq M(A)$. It follows from Lemma 4.6 that $QM(A) = LM(A) + RM(A)$, since B has an identity and $C(x) \otimes e_{11}$ is abelian. Let

$$I = \{ \{(a_{ij}^{(n)})\}_{n=1}^\infty : (a_{ij}^{(n)}) = 0, \text{ if } n \neq 1, (a_{ij}^{(n)}) \in B_0 \}.$$

Clearly I is an ideal of A . It follows from Theorem 6.3 that $QM(I) \neq LM(I) + RM(I)$, since $I \cong C(x) \otimes K$.

Example 8.3. There is a separable antiliminal C^* -algebra A such that $QM(A) \neq M(A)$, but $QM(A) = LM(A) + RM(A)$.

Let B be the nonelementary separable matroid C^* -algebra with identity obtained as the inductive limit of the following

$$M_{m(1)} \xrightarrow{g_1} M_{m(2)} \xrightarrow{g_2} M_{m(3)} \xrightarrow{g_3} \dots$$

where $g_i(x) = x \otimes p$ and $\dim p = m(2)/m(1)$ (see [10]). Let A_0 be the C^* -subalgebra of B generated by the elements a such that $a \in M_{m(k)}$ for some k , $a = (a_{ij})$, $a_{ij} = 0$, if $ij \neq 1$. Let A be the C^* -algebra of convergent sequences $\{a(n)\}$ in B with limits in A_0 .

(1) A is an antiliminal C^* -algebra. Let I be a nontrivial ideal of A and $I(k) = \{a(k) : a \in I\}$. There is a smallest integer k_0 such that $I(k_0) \neq \{0\}$. Clearly, $I(k_0)$ is an ideal of B . Since B is simple (see [10]), $I(k_0) = B$. Suppose $I_0 = \{a \in I : a(k_0) = 0\}$. Then I_0 is an ideal of I . Moreover $I/I_0 \cong I(k_0) = B$. Thus I is not liminal. So A is an antiliminal C^* -algebra.

(2) $\text{QM}(A) \neq \text{M}(A)$. Let x be the sequence such that $x(n) \in M_{m(k)}$ for some k and each n , moreover $(x(n))_{ij} = 1$ for all $i, j \leq m(k)$, and $x(\infty) = (a_{ij}^\infty)$, where $a_{11}^\infty = 1$, $a_{ij}^\infty = 0$, $ij \neq 1$. As in Example 8.1 and Example 8.2, one can easily check that $x \in \text{QM}(A)$, but $x \notin \text{M}(A)$.

(3) $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$. Since B has an identity, $\text{M}(B) = \text{QM}(B) = B$. Moreover A_0 is abelian, so $\text{M}(A_0) = \text{QM}(A_0)$. It follows from Lemma 4.6 that $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$.

9. THE DENSITY OF $\text{LM}(A) + \text{RM}(A)$ IN $\text{QM}(A)$

We know that $\text{QM}(A) \neq \text{LM}(A) + \text{RM}(A)$, in general. But is $\text{LM}(A) + \text{RM}(A)$ dense in $\text{QM}(A)$ in a suitable topology? (See [6, 7.2].)

Example 9.1. $\text{LM}(A) + \text{RM}(A)$ may not be norm closed.

Let X be the one-point compactification of the disjoint union of Y_n , $n = 1, 2, \dots$. Let $A = C(X, K)$. Use the same notations in the proof of Theorem 6.3. Define

$$F(t) = F_n(t)/\alpha(A_n)^{1/2}, \quad \text{if } t \in Y_n, \quad F(\infty) = 0.$$

As in the proof of Lemma 4.7, we see that $F \in \text{QM}(A)$, but $F \notin \text{LM}(A) + \text{RM}(A)$. Let $G_m(t) = F(t)$, if $t \in Y_n$, $n \leq m$, $G_m(t) = 0$, if $t \in Y_n$, $n > m$. Clearly $G_m \in \text{LM}(A) + \text{RM}(A)$ and $\|G_m(t) - F(t)\| \leq 1/\alpha(A_m)^{1/2} \rightarrow 0$, as $m \rightarrow \infty$. Hence $\text{LM}(A) + \text{RM}(A)$ is not norm closed.

Proposition 9.2. Let X be the disjoint union of Y_n , $n = 1, 2, \dots$, and take $A = C_0(X, K)$. Then $\text{LM}(A) + \text{RM}(A)$ is not norm dense in $\text{QM}(A)$.

Proof. Let $A_n = C(Y_n, K) \cong C(Y_n) \otimes K$. Take $x^{(n)} \in \text{QM}(A_n)$ such that $\|x^{(n)}\| \leq 1$ and $\alpha(x^{(n)}) \geq \alpha(C(Y_n, K)) - 1/n$. Define $x(t) = x^{(n)}(t)$ if $t \in Y_n$. Assume that $u = y + z$, such that $y \in \text{LM}(A)$, $z \in \text{RM}(A)$ and

$$\|x - u\| < 1/16.$$

Suppose $u = u^{(n)}(t)$, $t \in Y_n$, $y = y^{(n)}(t)$, $t \in Y_n$ and $z = z^{(n)}(t)$, $t \in Y_n$, $n = 1, 2, \dots$. Choose an integer N such that

$$\alpha(A_N) \geq \max(16, 16a),$$

where $a = \max(\|y\|, \|z\|)$. Suppose $x^{(N)} = y_1^{(N)} + z_1^{(N)}$ and $x^{(N)} - u^{(N)} = y_2^{(N)} + z_2^{(N)}$ such that $y_1^{(N)}, y_2^{(N)} \in \text{LM}(A)$, $z_1^{(N)} \in \text{RM}(A)$ and $\|y_2^{(N)}\| \leq (1/16)(\alpha(A_N) + 1/21)$.

Let $\{e_n\}$ be an approximate identity for A satisfying $e_m e_n = e_n e_m = e_n$, if $m > n$. By the proof of Lemma 2.1 and Theorem 2.3, there exists $n_1 < n_2 < \dots$ such that

$$\left\| \sum_{k=1}^{\infty} (1 - e_{n_{k+1}}) y (e_{n_k} - e_{n_{k-1}}) \right\| < \frac{1}{12},$$

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} (1 - e_{n_{k+1}}) z^* (e_{n_k} - e_{n_{k-1}}) \right\| &< \frac{1}{12}, \\ \left\| \sum_{k=1}^{\infty} (1 - e_{n_{k+1}}) y_i^{(N)} (e_{n_k} - e_{n_{k-1}}) |_{Y_N} \right\| &< \frac{1}{12}, \\ \left\| \sum_{k=1}^{\infty} (1 - e_{n_{k+1}}) (z_i^{(N)})^* (e_{n_k} - e_{n_{k-1}}) |_{Y_N} \right\| &< \frac{1}{12}, \end{aligned}$$

$i = 1, 2$, and

$$\sum_{k=1}^{\infty} (1 - e_{n_k}) x^{(N)} (e_{n_k} - e_{n_{k-1}}) |_{Y_N} \in \text{RM}(A_N).$$

Thus

$$\left\| \sum_{k=1}^{\infty} (1 - e_{n_k}) x^{(N)} (e_{n_k} - e_{n_{k-1}}) |_{Y_N} \right\| \geq \alpha(A_N) - 1 - \frac{1}{N}.$$

By the proof of Lemma 4.3,

$$\left\| \sum_{k=1}^{\infty} (1 - e_{n_k}) u (e_{n_k} - e_{n_{k-1}}) \right\| \leq 7a + 4,$$

and

$$\left\| \sum_{k=1}^{\infty} (1 - e_{n_k}) (x^{(N)} - u^{(N)}) (e_{n_k} - e_{n_{k-1}}) |_{Y_N} \right\| \leq \frac{1}{16} (7\alpha(A_N) + 6) \leq \frac{1}{2} \alpha(A_N).$$

But

$$\frac{1}{2} \alpha(A_N) + 7a + 4 < \alpha(A_N) - 1 - \frac{1}{n}.$$

A contradiction. Hence

$$\|x - u\| \geq 1/16.$$

Theorem 9.3. *Let A be a C^* -algebra. Then $\text{LM}(A) + \text{RM}(A)$ is strictly dense in $\text{QM}(A)$. Moreover, for every $x \in \text{QM}(A)$, there is a net $\{x_\lambda\} \subset \text{LM}(A) + \text{RM}(A)$ such that $\|x_\lambda\| \leq 2\|x\|$ and $x_\lambda \rightarrow x$ strictly. If A is σ -unital, $\{x_\lambda\}$ can be taken as a sequence.*

Proof. Take $x \in \text{QM}(A)$ with $\|x\| \leq 1$. Let $\{e_\lambda\}$ be an approximate identity for A . Define $x_\lambda = e_\lambda x (1 - e_\lambda) + x e_\lambda$. Clearly $e_\lambda x (1 - e_\lambda) \in \text{LM}(A)$, $x e_\lambda \in \text{RM}(A)$.

For every $\varepsilon > 0$ and $a \in A$, there is λ_0 such that if $\lambda \geq \lambda_0$, then $\|a(1 - e_\lambda)\| < \varepsilon/2$ and $\|(1 - e_\lambda)a\| < \varepsilon/2$. Thus

$$\begin{aligned} \|a(x_\lambda - x)\| &= \|a e_\lambda x (1 - e_\lambda) - a x (1 - e_\lambda)\| \\ &\leq \|a e_\lambda - a\| \|x(1 - e_\lambda)\| < \varepsilon/2 < \varepsilon, \end{aligned}$$

and

$$\begin{aligned} \|(x_\lambda - x)a\| &= \|e_\lambda x (1 - e_\lambda)a + x e_\lambda a - x a\| \\ &\leq \|e_\lambda x\| \|(1 - e_\lambda)a\| + \|x\| \|(e_\lambda - 1)a\| < \varepsilon. \end{aligned}$$

Moreover $\|x_\lambda\| \leq 2$. If A is σ -unital, $\{e_\lambda\}$ can be taken as a sequence, so $\{x_\lambda\}$ is a sequence.

Let X be the disjoint union of Y_n , $n = 1, 2, \dots$, and take $A = C_0(X) \otimes K$. It follows from Theorem 6.3 that $\text{QM}(A) \neq \text{LM}(A) + \text{RM}(A)$. However, for every $x \in \text{QM}(A)$, if we define $x_n(t) = x(t)$ for $t \in Y_m$ and $m \leq n$, $x_n(t) = 0$ for $t \in Y_m$ and $m > n$, then $x_n \in \text{LM}(A) + \text{RM}(A)$ (Lemma 4.7), and $\|\pi^{**}(x_n) - \pi^{**}(x)\| \rightarrow 0$ uniformly on every compact subset of $\hat{A}$, with $\|x_n\| \leq \|x\|$. This type of density is stronger than the strict density considered in Theorem 9.3. Indeed, if $a \in A$, then $C = \{\pi \in \hat{A}, \|\pi(a)\| \geq \varepsilon\}$ is a compact subset of $\hat{A}$. Thus there is N such that

$$\|\pi(a)[\pi^{**}(x_n) - \pi^{**}(x)]\| < \varepsilon, \quad \pi \in C,$$

and

$$\|\pi(a)[\pi^{**}(x_n) - \pi^{**}(x)]\| < \varepsilon \cdot 2\|x\|,$$

if $\pi \in \hat{A} \setminus C$. From these inequalities, we see that $x_n \rightarrow x$ strictly. The construction of x_n depends largely on the fact that $\hat{A}$ is Hausdorff. If X is a countable locally (quasi-) compact space with $\lambda(X) \leq \infty$, we say X satisfies condition (C), if for every $t \in X \setminus X_{[\infty]}$ there is an open set O_t such that $t \in O_t$ and $\overline{O_t} \cap X_{[k]} = \emptyset$ for some k . Clearly, if X is Hausdorff, then X satisfies condition (C). If each point in $X \setminus X_{[\infty]}$ has a clopen neighborhood, then X also satisfies condition (C).

Theorem 9.4. *Let A be a separable C^* -algebra with countable spectrum $\hat{A}$ and $\hat{A}_{[\infty]} = \emptyset$. If $\hat{A}$ satisfies condition (C), then for every $x \in \text{QM}(A)$, there is a sequence $\{y_n\} \subset \text{LM}(A) + \text{RM}(A)$ such that $\|y_n\| \leq 3\|x\|$ and $\pi^{**}(y_n) = \pi^{**}(x)$ eventually on every compact subset of $\hat{A}$.*

Proof. Take $x \in \text{QM}(A)$ with $\|x\| \leq 1$. Put $I_n = \{a \in A: \pi(a) = 0, \forall \pi \in \hat{A}_{[n]}\}$, $n = 1, 2, \dots$. Let $\{e_i\}$ be an approximate identity for A and $\{p_m^n\}_{m=1}^\infty$ be an approximate identity for I_n . Define

$$x_{ij} = (e_i - e_{i-1})^{1/2} x (e_j - e_{j-1})^{1/2}.$$

Thus $x_{ij} \in A$, and since the norm closure $\bigcup_n I_n$ is A , we can find $\{p_i\} \subset \{p_m^n, m, n = 1, 2, \dots\}$ satisfying:

$$\|x_{ij}(1 - p_j)\| < \frac{1}{2^{i+j}}, \quad i \leq j,$$

and

$$\|(1 - p_i)x_{ij}\| < \frac{1}{2^{i+j}}, \quad j \leq i.$$

Define $p = \sum_{i=1}^\infty (e_i - e_{i-1})^{1/2} p_i (e_i - e_{i-1})^{1/2}$. Clearly $p \in \text{M}(A)$. By Lemma 2.1, we see that $(1 - p)xp + x(1 - p) \in \text{LM}(A) + \text{RM}(A)$ (as in the proof of Lemma 3.2). Without loss of generality, we may assume that $p_i \in I_i$.

Let $\widehat{A} = \{\pi_1, \pi_2, \dots\}$. Fix n , and let O_n be an open set of $\widehat{A}$ such that $\pi_1, \pi_2, \dots, \pi_n \in O_n$ and $\overline{O_n} \cap \widehat{A}_{[k]} = \emptyset$ for some k . This is possible since $\widehat{A}$ satisfies condition (C). Moreover, we may assume that $O_n \subset O_{n+1}$.

Let $J_n = \{a \in A; \pi(a) = 0, \forall \pi \in \overline{O_n}\}$. Clearly, if ϕ_n is the canonical homomorphism from A to A/J_n , then $\phi_n(I_k) = \phi(A)$. Let q_i be an element in I_k such that $\|q_i\| \leq 1$ and $\phi_n(q_i) = \phi_n(p_i)$. Thus $\pi(q_i) = \pi(p_i)$ if $\pi \in \overline{O_n}$. Define

$$q^{(n)} = \sum_{i=1}^{\infty} (e_i - e_{i-1})^{1/2} q_i (e_i - e_{i-1})^{1/2}.$$

Then $q^{(n)} \in \text{M}(A, I_k)$. Put $z_n = q^{(n)} x q^{(n)}$. Then $z_n \in \text{QM}(A, I_k)$. It follows from Lemma 6.1 that $\text{QM}(I_k) = \text{LM}(I_k) + \text{RM}(I_k)$. By Lemma 3.2, $z_n \in \text{LM}(A, I_k) + \text{RM}(A, I_k) \subset \text{LM}(A) + \text{RM}(A)$. Define $y_n = (1-p)xp + x(1-p) + z_n$. Clearly $y_n \in \text{LM}(A) + \text{RM}(A)$. Moreover, $\|y_n\| \leq 3\|x\|$.

Let S be a compact subset of $\widehat{A}$, $S = \{\pi_1, \pi_2, \dots\}$. We have $\bigcup_n O_n \supset S$. Thus there are $n_1, n_2, \dots, n_m$ such that $\bigcup_{j=1}^m O_{n_j} \supset S$. Since $O_n \subset O_{n+1}$, there is an integer N , such that $O_N \supset S$. If $n \geq N$, $\pi^{**}(z_n) = \pi^{**}(p x p)$ for $\pi \in \overline{O_N}$. Thus $\|\pi^{**}(y_n) - \pi^{**}(x)\| = 0$ if $\pi \in S$.

Theorem 9.5. *Let A be a separable C^* -algebra of type I. Suppose that there is an integer N such that for every $\pi \in \widehat{A}$, the closure $\{\pi\}^-$ of $\{\pi\}$ is countable and $\lambda(\{\pi\}^-) \leq N$. Then for every $x \in \text{QM}(A)$, there is a bounded net $\{x_\alpha\} \subset \text{LM}(A) + \text{RM}(A)$ such that for every $\pi \in \widehat{A}$*

$$\lim \|\pi^{**}(x_\alpha) - \pi^{**}(x)\| = 0$$

and $x_\alpha \rightarrow x$ strictly.

Proof. Let Γ be the family of finite subsets of $\widehat{A}$. Fix $\alpha \in \Gamma$. Then α^- is countable. Moreover, $\lambda(\alpha^-) \leq \max\{\lambda(\{\pi\}^-), \pi \in \alpha\} \leq N$.

We may assume that $\|x\| \leq 1$. Let $J_\alpha = \bigcap_{\pi \in \alpha} \ker \pi$. Then $(A/J_\alpha)^\wedge$ is countable and $\lambda[(A/J_\alpha)^\wedge] \leq N$. Let $\phi: A \rightarrow A/J_\alpha$ be the canonical homomorphism from A to A/J_α . It follows from the proof of Lemma 6.1 that there are $\bar{y}'_\alpha \in \text{LM}(A/J_\alpha)$, $\bar{y}''_\alpha \in \text{RM}(A/J_\alpha)$ such that $\phi(x) = \bar{y}'_\alpha + \bar{y}''_\alpha$, $\|\bar{y}'_\alpha\| \leq 3^N$ and $\|\bar{y}''_\alpha\| \leq 3^N$. It follows from [6] that there are $y_\alpha \in \text{LM}(A) + \text{RM}(A)$ such that $\phi(y_\alpha) = \bar{y}'_\alpha + \bar{y}''_\alpha = \phi(x)$ and $\|y_\alpha\| \leq 2 \cdot 3^N$. Let $z_\alpha = x - y_\alpha$, then $\|z_\alpha\| \leq 2 \cdot 3^N + 1$. Suppose that $\{e_n\}$ is an approximate identity for A . Define

$$u_\alpha = e_{|\alpha|} z_\alpha (1 - e_{|\alpha|}) + z_\alpha e_{|\alpha|}$$

and $x_\alpha = y_\alpha + u_\alpha$. Clearly $x_\alpha \in \text{LM}(A) + \text{RM}(A)$ and $\|x_\alpha\| \leq 4 \cdot 3^N + 2$. It is easy to check that

$$\|\pi^{**}(x_\alpha) - \pi^{**}(x)\| \rightarrow 0$$

for every $\pi \in \widehat{A}$. Moreover, since $x - x_\alpha = z_\alpha - u_\alpha$, by the proof of Theorem 9.3, we have $x_\alpha \rightarrow x$ strictly.

Corollary 9.6. *Let A be a separable liminal C^* -algebra. Then for every $x \in \text{QM}(A)$, there is a bounded net $\{x_\lambda\} \subset \text{LM}(A) + \text{RM}(A)$ such that for every $\pi \in \hat{A}$*

$$\lim \|\pi^{**}(x_\lambda) - \pi^{**}(x)\| = 0$$

and $x_\alpha \rightarrow x$ strictly.

Proof. $\hat{A}$ is a T_1 space.

Note. The problem $\text{QM}(A) = \text{LM}(A) + \text{RM}(A)$ for simple C^* -algebras has been studied and the results will appear elsewhere.

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